MATHCORE: Exercises 19/07/2018

Problem [1]. Approximation of the image: `saturn.png`. After starting `MATLAB`, type

```
saturn=imread('saturn.png');
saturn0=double(saturn)/256; saturn0=saturn0(:,:,1);
Z=saturn0;
[mz,nz] = size(Z);
imshow(Z,[0 1]);
mag = 1/3;
truesize(1, [mz*mag, nz*mag]);
```

Compute the SVD of this image, using the command: `[U,S,V] = svd(Z);`

1. Plot the singular values on a logarithmic scale.
2. Compute the optimal approximants in the 2-norm having rank $r = 1, 5,$ and the optimal approximant having 2-norm error less than 2%, of the largest singular value of Z. What is the rank of the approximant in the latter case? For each of the three cases compute the compression ratio. ■

Problem [2]. Ellipse fitting using LS and TLS.

Consider 63 points in 2-dimensional space, constructed as follows:

```
>> th=0:.1:2*pi;
>> a=1;b=2;
>> x=a*cos(th); y=b*sin(th);
>> figure; plot(x,y); hold;
>> xn=x+0.08*randn(1,63);
>> yn=y+0.08*randn(1,63);
>> plot(xn,yn,'r*')
>> % Comment: you should now see an ellipse in blue and
>> % red dots defining an approximate (noisy) ellipse.
>> % The goal is to fit to the red dots an ellipse using
>> % LS and TLS and compare it with the original ellipse.
```

Recall that the ellipse above is defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. To approximate this equation in LS we seek coefficients α and β such that the sum of the squares of the distances

$$\delta(k,1) = [xn(k,1)^2, \quad yn(k,1)^2] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - 1, \quad i = 1, 2, \cdots, 63,$$

is minimized, i.e. $\sum_{i=1}^{63} \delta^2(i,1)$, is minimized. For the TLS approximation we seek $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, such that the sum of the squares of

$$\epsilon(k,1) = [xn(k,1)^2, \quad yn(k,1)^2, \quad 1] \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix}, \quad i = 1, 2, \cdots, 63,$$

i.e. $\sum_{i=1}^{63} \epsilon^2(i,1)$, is minimized.

(a) Set up the above problem in the LS, TLS, frameworks respectively, and solve is. Then, find the error in each case. How do the coefficients α, β and $\frac{\bar{\alpha}}{\bar{\gamma}}$, $\frac{\bar{\beta}}{\bar{\gamma}}$, compare with the noiseless values of a and b?

(b) Draw the two ellipses obtained, together with the original data points. ■

Problem [3]. 1. **Power Method for computing the PageRank**.

In this problem, we will use the Power Method for ranking MPI-related webpages.

(a) (20pts) Construct a database of webpages starting from an initial URL. For this task, we will use the m-file surfer.m of Cleve Moler (founder of MATLAB). (This m-file can be found by searching the MathWorks website.

Type the following statement to create your database of webpages:

$$[U, G] \;=\; \text{surfer('http://www.mpi-magdeburg.mpg.edu', 1000);}$$

The code starts at the specified URL (http://www.mpi-magdeburg.mpg.edu in this case), and surfs the Web until it has visited $n = 1000$ pages. Be patient, as this might take a while and the code may freeze. In this case, you will need to restart the search. Once completed, the surfer.m function will return an $n \times 1$ cell array U of URLs, and an $n \times n$ sparse connectivity matrix G. You can look at the entires of U to find out which webpages were visited. The matrix G shows how the webpages are linked to each other. Visualize the structure of this connectivity matrix by using the command spy(G).

(b) Next, we construct the transition matrix A for the Markov process, which views Web surfing as a random process $x(t+1) = Ax(t)$, $t \in \mathbb{Z}$, where the k^{th} entry of $x \in \mathbb{R}^n$ gives the probability that, in the long run, a random surfer will end up at the k^{th} Web site. Perform the following commands to create the matrix A. Note that this code uses the probability value $p = 0.85$.

```
n = size(G,2);
p = 0.85; % The probability we used in the class
delta = (1-p)/n;
c = sum(G,1);
k = find(c~=0);
D = sparse(k,k,1./c(k),n,n);
e = ones(n,1);
z = ((1-p)*(c~=0) + (c==0))/n;
A = p*G*D + e*z; % This corresponds to p*M + (1-p)/n*e*e' discussed in class
```

(c) We know that the PageRank corresponds to entries of the right eigenvector v corresponding to the dominant eigenvalue 1 of A. Run the Power Method with the initial vector

$$v^{(0)} = \text{ones}(n, 1)/n,$$

corresponding to initially equal probabilities. In the Power Method implementation, normalize the approximate eigenvectors by the 2-norm. In this case, since A is a stochastic matrix and v_0 is a probability vector, the approximate eigenvectors are also probability vectors, throughout the iteration; thus there is no need to normalize them separately. For a stopping criterion, simply check the 2-norm distance between $v^{(k)}$ and $v^{(k-1)}$, i.e, the 2-norm distance between two consecutive approximate eigenvectors. Run the iteration until this distance is below 10^{-4}. My implementation converged after about 30 steps. What are the top five webpages in this PageRanking problem and what are the probabilities of being visited?

Hint: it is recommended that you solve the above problems first for a small n (say, $n = 20 \sim 40$), where there is no need for iterative methods for computing the eigenvalue decomposition of the google matrix. ■

Problem [4]. **(a)** Given is a matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$, with eigenvalues $\lambda_1, \cdots, \lambda_n$. Choose among these a *preferred* subset $\mu_1 = \lambda_{i_1}, \cdots, \mu_k = \lambda_{i_k}$, for appropriate indices $i_1, \cdots, i_k$. Assume for simplicity that $i_1 = 1, \cdots, i_k = k$, i.e. the preferred eigenvalues are the first k ones. Let

$$\mathbf{A} = \mathbf{V\Lambda V^{-1}},$$

be the EVD of $\mathbf{A}$. We partition the eigenvectors to preferred ones (consisting of the columns of) $\mathbf{V_1}$, and the rest:

$$\mathbf{V} = [\mathbf{V_1} \quad \mathbf{V_2}], \quad \text{where} \quad \mathbf{V_1} \in \mathbb{C}^{\mathbf{n \times k}}, \quad \mathbf{V_2} \in \mathbb{C}^{\mathbf{n \times (n-k)}}.$$

Let the columns of $\mathbf{W_1} \in \mathbb{C}^{\mathbf{n \times (n-k)}}$ constitute an orthonormal basis for the orthogonal complement of $\mathbf{V_1}$, in other words $\mathbf{W_1^* V_1} = \mathbf{0}$ and $\mathbf{W_1^* W_1} = \mathbf{I_{n-k}}$.

Show that the eigenvalues of $\mathbf{W_1^* A W_1}$ are $\lambda_{k+1}, \cdots, \lambda_n$. Notice that $\mathbf{W_1}$ can be computed by means of the SVD of $\mathbf{V_1}$. **This operation is sometimes referred to as deflation of the eigenvalues of A.**

(b) Use this result to compute the second largest eigenvalue of the Google matrix by means of the power iteration method. ■

Problem [5]. Here we start by plotting some relevant quantities describing our system. If you are familiar with system poles, system stability, frequency response and impulse response, you may skip this problem.

(a) Compute the *system poles* (eigenvalues of $\mathbf{A}$) using the command `eig(A)` and plot them in the complex plane. Is the system stable, i.e. are all poles in the open left-half plane? Why are the system poles in complex conjugate pairs?

(b) Next, produce a `loglog` plot of the system's *frequency response*, that is, plot the magnitude of the transfer function $\mathbf{H}(\jmath\omega) = \mathbf{C}\big(\jmath\omega\mathbf{E} - \mathbf{A}\big)^{-1}\mathbf{B} + \mathbf{D}$ for 200 frequency points $\omega \in [10^{-2}, 10^3]$.
Notice that for our clamped beam system $\mathbf{E} = \mathbf{I}$ and $\mathbf{D} = \mathbf{0}$. Explain why $|\mathbf{H}(\jmath\omega)| \to 0$ as $\omega \to \infty$.

(c) For the case $\mathbf{E} = \mathbf{I}$, the *impulse response* of a system is defined as $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t), t \geq 0$, and represents the system output $\mathbf{y}(t)$ when the input $\mathbf{u}(t) = \delta(t)$.
Use the command `expm()` to compute the matrix exponential and then plot the impulse response for an equally spaced time grid `t = linspace(0,500,500)`. (*Hint:* You only need to compute the matrix exponential once.) Explain why the impulse response $\mathbf{h}(t) \to 0$ as $t \to \infty$.

(d) Next, compute the output $\mathbf{y}(t)$ for a general input $\mathbf{u}(t)$. Recall the inpus-state-output relationship

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \end{aligned}$$

Approximating $\dot{\mathbf{x}}(t)$ with $(\mathbf{x}(t) - \mathbf{x}(t - \Delta t))/\Delta t$, where Δt is a small, constant time step, it immediately follows that one can write the state at time t as $\mathbf{x}(t) = (\mathbf{E} - \Delta t\mathbf{A})^{-1}(\mathbf{E}\mathbf{x}(t - \Delta t) + \Delta t\mathbf{B}\mathbf{u}(t))$. We can now subsitute this expression for $\mathbf{x}(t)$ into the output equation to compute $\mathbf{y}(t)$ for general $\mathbf{u}(t)$. This procedure is often refered to as *Backward Euler*.
Asumming $\mathbf{x}(0) = \mathbf{0}$, plot the output $\mathbf{y}(t)$ for the input `u = ones(1,2000), u(1:1000) = -1`, and time grid `t = linspace(0,1000,2000)`.

■

Problem [6]. (Optimal $\mathcal{H}_2$ model reduction) Run the routine `irka_pseudocode_with_E.m` to construct matrices $\mathbf{W}_k \in \mathbb{R}^{n \times k}$ and $\mathbf{V}_k \in \mathbb{R}^{n \times k}$, and then compute the reduced order model

$$\mathbf{A}_k = \mathbf{W}_k^* \mathbf{A} \mathbf{V}_k, \quad \mathbf{B}_k = \mathbf{W}_k^* \mathbf{B}, \quad \mathbf{C}_k = \mathbf{C}\mathbf{V}_k.$$

The impulse response $\mathbf{h}_k(t) = \mathbf{C}_k e^{\mathbf{A}_k t}\mathbf{B}_k$ of the IRKA reduced model minimizes the $\mathcal{H}_2$ norm of the error system, that is:

$$\left(\int_{-\infty}^{\infty} (\mathbf{h} - \mathbf{h}_k)^2(t)dt\right)^{1/2} \leq \left(\int_{-\infty}^{\infty} (\mathbf{h} - \widehat{\mathbf{h}}_k)^2(t)dt\right)^{1/2},$$

for any reduced model of order k with impulse responses $\widehat{\mathbf{h}}_k$.

For this problem, take $k = 10$.

(a) Take the initial shift selection `S = 1j*logspace(-2,2,k).'` and then run IRKA with the settings

```
E = eye(length(B));
Tol1 = 1e-8;
Tol2 = 1e-8;
MAXITER = 100;
[Ak,Ek,Bk,Ck,Siter]=irka_pseudocode_with_E(A,E,B,C,S,Tol1,Tol2,MAXITER);
```

Notice that IRKA produces a reduced model in *descriptor* state-space form, i.e. the $\mathbf{E}$ matrix may not be the identity. Compare the frequency response of the IRKA reduced model $\mathbf{H}_k(\jmath\omega) = \mathbf{C}_k\left(\jmath\omega\mathbf{E}_k - \mathbf{A}_k\right)^{-1}\mathbf{B}_k + \mathbf{D}$ with the frequency response of the full order system from Problem [1](b).

(b) Compare the impulse response $\mathbf{h}_k(t)$ of the reduced model with the impulse response $\mathbf{h}(t)$ of the full order system from the previous Problem.

Notice that the impulse response formula $\mathbf{h}_k(t) = \mathbf{C}_k e^{\mathbf{A}_k t}\mathbf{B}_k + \mathbf{D}\delta(t)$ holds for the case when $\mathbf{E}_k = \mathbf{I}_k$. To transform our IRKA reduced model to a system with $\mathbf{E}_k = \mathbf{I}_k$, we define

$$\widehat{\mathbf{A}}_k = \mathbf{E}_k^{-1}\mathbf{A}_k, \quad \widehat{\mathbf{B}}_k = \mathbf{E}_k^{-1}\mathbf{B}_k,$$

and then the impulse response of the reduced model is given by $\mathbf{h}_k(t) = \mathbf{C}_k e^{\widehat{\mathbf{A}}_k t}\widehat{\mathbf{B}}_k + \mathbf{D}\delta(t)$.

(c) Repeat (a) and (b) for $k = 10$, but with random initial shift selection `S = 1j*randn(k,1)`.

Does IRKA converge? If so, in how many iterations? Plot the frequency and impulse responses of the reduced model (even if IRKA didn't fully converge)? What do you notice?

■

Problem [7]. (Reduced models from measurements (Loewner method)

Download the file `freq_data.mat` and use the `load` command to load its contents into MATLAB.

The vector $w \in \mathbb{C}^N$ contains $N = 100$ points on the $\jmath\omega$ axis, $w(i) \in \jmath\,[0, \infty)$, and the entries of $H \in \mathbb{C}^N$ are frequency response measurements computed as

$$H(i) = \mathbf{C}\big(w(i)\mathbf{E} - \mathbf{A}\big)^{-1}\mathbf{B}$$

for an *unknown* system $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times 1}, \mathbf{C} \in \mathbb{R}^{1\times n}$, of unknown order n.

Our goal is to find a reduced model of order k, given by $\mathbf{E}_k, \mathbf{A}_k \in \mathbb{R}^{k\times k}, \mathbf{B}_k \in \mathbb{R}^{k\times 1}, \mathbf{C}_k \in \mathbb{R}^{1\times k}$, that *interpolates* the given measurements

$$\mathbf{H}_k\big(w(i)\big) = H(i)$$

with $\mathbf{H}_k(s) = \mathbf{C}_k\left(s\mathbf{E}_k - \mathbf{A}_k\right)^{-1}\mathbf{B}_k$.

To identify the low order k and construct matrices $\mathbf{E}_k, \mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k$, we use the *Loewner matrix* framework.

(a) Partition the given measurements into any two disjoint sets

$$\big\{(w_i, H_i)\big\} = \big\{(\lambda_j, W_j)\big\} \cup \big\{(\mu_k, V_k)\big\},$$

for $j = 1, \dots, M$ and $k = 1, \dots, N - M$. For simplicity, take $M = N/2$, and define the partitioning

```
la = w(1:2:N);          W = H(1:2:N);
mu = w(2:2:N);          V = H(2:2:N);
```

Next, to ensure that our reduced models are real, i.e. matrices $\mathbf{E}_k, \mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k$ have real entries, we also need to use the complex conjugate values of the measurements. Thus, define vectors

$$\lambda \leftarrow [\lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2, \cdots], \qquad W \leftarrow [W_1, \overline{W}_1, W_2, \overline{W}_2, \cdots],$$

$$\mu \leftarrow [\mu_1, \overline{\mu}_1, \mu_2, \overline{\mu}_2, \cdots], \qquad V \leftarrow [V_1, \overline{V}_1, V_2, \overline{V}_2, \cdots].$$

From these measurements, form the *Loewner* matrix $\mathbb{L}$ and *shifted-Loewner* matrix $\mathbb{L}_s$ defined as

$$\mathbb{L}(i,j) = \frac{V_i - W_j}{\mu_i - \lambda_j}, \qquad \mathbb{L}_s(i,j) = \frac{\mu_i V_i - \lambda_j W_j}{\mu_i - \lambda_j}.$$

Notice that the Loewner matrices still have complex entries. To obtain matrices with real entries, apply the following transformation

$$\mathbb{L} \leftarrow \mathbf{P}^*\mathbb{L}\mathbf{P}, \qquad \mathbb{L}_s \leftarrow \mathbf{P}^*\mathbb{L}_s\mathbf{P},$$

$$W \leftarrow (W^T\mathbf{P})^T, \qquad V \leftarrow \mathbf{P}^*V,$$

where $\mathbf{P}$ is a block-diagonal matrix, with each diagonal block given by the matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \jmath \\ 1 & -\jmath \end{bmatrix}$.

Then, the order k of the reduced order interpolant $\mathbf{H}_k$ is given by the rank of $\mathbb{L}$.

Plot the singular values of $\mathbb{L}$ and identify k as the numerical rank of $\mathbb{L}$.

(b) Let $\mathbb{L} = \mathbf{YSX}^*$ be the singular value decomposition (SVD) of the Loewner matrix $\mathbb{L}$. Define

```
Yk = Y(:,1:k);
Xk = X(:,1:k);
```

Then the reduced order model $\mathbf{H}_k$ is given by

$$\mathbf{E}_k = -Y_k^*\mathbb{L}X_k, \quad \mathbf{A}_k = -Y_k^*\mathbb{L}_sX_k, \quad \mathbf{B}_k = Y_k^*V, \quad \mathbf{C}_k = W^TX_k.$$

Produce a plot superimposing the given measurements (w_i, H_i) over the values obtained by evaluating the reduced order model at the same points w_i, i.e. $\mathbf{H}_k\big(w(i)\big)$. Does your reduced order model of order k interpolate the given measurements?

Remark: Notice that the given points w_i are all on the $\jmath\omega$ axis. Therefore, we can easily produce frequency response plots with the command `loglog(imag(w),abs(H))`, where the $0x$ axis represents radians/second. If we normalize w by 2π, then this axis will be given in Hertz.

(c) As some of you may have already noticed, the file `freq_data.mat` contains frequency response measurements of the clamped beam system discussed in previous problems.

Load the original system $\mathbf{A}, \mathbf{B}, \mathbf{C}$ of order $n = 348$ stored in `beam.mat`. Superimpose its frequency response over the plot from point (b) for a fairly dense frequency grid, e.g. `logspace(-2,3,300)`.

How well does your reduced model from (b) approximate the original system?

(d) Next, on the same plot, show the poles of the original system, i.e. the eigenvalues of $\mathbf{A}$, together with the poles of the reduced model you constructed in (b). The poles of your reduced model are the *generalized* eigenvalues of the matrix pencil $(\mathbf{A}_k, \mathbf{E}_k)$. How do the poles of the reduced model compare with those of the original system? Is your reduced order model stable?

(e) Repeat the experiments in points (b), (c) and (d) for smaller reduced orders, e.g. $k = 10$. Are the given measurements (w_i, H_i) interpolated by your reduced order models? How well is the frequency response of the original system $\mathbf{A}, \mathbf{B}, \mathbf{C}$ approximated?

(f) On the same plot, show the impulse response of the full order system, the reduced model from point (b) ($k = \operatorname{rank} \mathbb{L}$) and the reduced model from point (e) ($k = 10$).

Further details on the Loewner matrix framework for rational interpolation:

- Rational interpolation and system identification in the Loewner matrix framework:

  A.J. Mayo and A.C. Antoulas, *A framework for the generalized realization problem*, Linear Algebra and Its Applications, vol. 425: 634-662 (2007).

- The general case of multi-input multi-output (MIMO) system measurements, e.g. S-parameters:

  S. Lefteriu and A.C. Antoulas, *A new approach to modeling multi-port systems from frequency domain data*, IEEE Transactions on Computer Aided Design of Integrated Circuits and Systems, vol. 29, no. 1, pp. 14-27, Jan. 2010.

■

Problem [8]. (Balanced truncation) Next, we compute reduced order models using *balanced truncation*.

(a) Load the system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ stored in `beam.mat`. In MATLAB, use `lyapchol` to compute the Cholesky factors for the controllability gramian $\mathbf{P}$ and observability gramian $\mathbf{Q}$, i.e. $\mathbf{P} = \mathbf{U}\mathbf{U}^*$ and $\mathbf{Q} = \mathbf{L}\mathbf{L}^*$, where $\mathbf{U}$ is upper triangular and $\mathbf{L}$ is lower triangular. Recall that the gramians are the solutions of the following Lyapunov equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^* + \mathbf{B}\mathbf{B}^* = \mathbf{0} \qquad \text{and} \qquad \mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0}.$$

(b) Compute the singular value decomposition

$$\mathbf{U}^*\mathbf{L} = \mathbf{Y}\Sigma\mathbf{X}^*$$

Then the *Hankel singular values* of the system (square roots of the eigenvalues of the product $\mathbf{PQ}$) are given by the diagonal entries σ_i of Σ. Produce a `semilogy` plot showing the eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$ together with the Hankel singular values σ_i.

(c) Let the order of the desired reduced model be k and partition the singular value decomposition as

$$\mathbf{Y}\Sigma\mathbf{X}^* = \left[\mathbf{Y}_k \widehat{\mathbf{Y}}_k\right] \begin{bmatrix} \Sigma_k & \\ & \widehat{\Sigma}_k \end{bmatrix} \begin{bmatrix} \mathbf{X}_k^* \\ \widehat{\mathbf{X}}_k^* \end{bmatrix}$$

where $\Sigma_k \in \mathbb{R}^{k\times k}$. Then the balanced truncation reduced order model is given by

$$\mathbf{A}_k = \mathbf{W}_k^*\mathbf{A}\mathbf{V}_k, \quad \mathbf{B}_k = \mathbf{W}_k^*\mathbf{B}, \quad \mathbf{C}_k = \mathbf{W}_k^*\mathbf{C},$$

where $\mathbf{W}_k = \mathbf{L}\mathbf{X}_k\Sigma_k^{-1/2}$ and $\mathbf{V}_k = \mathbf{U}\mathbf{Y}_k\Sigma_k^{-1/2}$.
For several values of k, produce frequency response plots comparing the original system with the reduced order models. Also, produce a plot showing the spectral abscissa of $\mathbf{A}_k$ as a function of k.

(d) For a fixed k, e.g. $k = 20$, produce a plot of the error between the original system and the balanced truncation reduced model, i.e. plot $|\mathbf{H}(\jmath\omega) - \mathbf{H}_k(\jmath\omega)|$. On the same plot, superimpose the balanced truncation upper error bound, i.e. $2(\sigma_{k+1} + \cdots + \sigma_n)$.

Problem [9]. Data-Driven Optimal Model Reduction: Parts (a) and (b) of this problem are from *Approximation of Large-scale Dynamical Systems*, A.C. Antoulas, SIAM Press, 2009.
Consider the diffusion of heat through a perfectly insulated, heat-conducting rod described by

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \quad t \geq 0, \quad x \in [0,1]$$

with the boundary conditions

$$\frac{\partial T}{\partial t}(0,t) = 0 \quad \text{and} \quad \frac{\partial T}{\partial x}(1,t) = u(t)$$

where $u(t)$ is the input function (supplied heat) and the output is $y(t) = T(0,t)$

(a) Show that the transfer function is given by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{\sqrt{s}\sinh\sqrt{s}}$$

(b) Show that the poles are $\lambda_k = -k^2\pi^2$, $k = 0, 1, 2, \ldots$ and

$$H(s) = \sum_{k=0}^{\infty} \frac{\phi_k}{s + k^2\pi^2}, \quad \phi_0 = 1, \quad \phi_k = (-1)^k\, 2, k > 0$$

(c) Construct a locally optimal $\mathcal{H}_2$ approximation using IRKA. We will do this without discretization by applying realization-independent IRKA using the Loewner framework for IRKA. Note that $H(s)$ has a pole at $s = 0$. Remove this pole and apply Loewner-based IRKA to obtain a second-order rational approximation. Then, add back the pole at zero to your rational approximation to get your final approximation. Note that this rational approximation is an *exact* Hermite interpolant to $H(s)$. Verify this numerically. Find the $\mathcal{H}_\infty$ error.

Problem [10]. RLC circuit - MIMO - Band-stop filter[1]**:** This problem described in *Data-driven and interpolatory model reduction*, S. Gugercin, C. A. Beattie, and A. C. Antoulas, SIAM Philadelphia, 2016. This system has 2 ports (i.e. two inputs and two outputs) state-space dimension 10, and a $\mathbf{D}$ term of size 2×2. The Linear Time Invariant (LTI) system in the matrix form is:

$$\mathbf{\Sigma} : \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t).$$

$$\mathbf{\Sigma} : (\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}), \quad \mathbf{E}, \mathbf{A} \in \mathbb{R}^{\mathbf{10 \times 10}}, \mathbf{B} \in \mathbb{R}^{\mathbf{10 \times 2}}, \mathbf{C} \in \mathbb{R}^{\mathbf{2 \times 10}}, \mathbf{D} \in \mathbb{R}^{\mathbf{2 \times 2}}$$

(a) In this case where the matrix pencil $(\mathbf{A}, \mathbf{E})$ is regular, show that the transfer function is defined as:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \in \mathbb{R}^{2 \times 2}.$$

(b) Show that we can eliminate the $\mathbf{D}$ term by incorporating it in the remaining matrices $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$. To achieve this, we have to allow the dimension of the realization to increase by $rank(\mathbf{D})$. Use the remark 1.1.1 on page 6 from: *A tutorial introduction to the Loewner Framework for Model Reduction*, A.C. Antoulas, S. Lefteriu, and A.C. Ionita. and show that the descriptor system

$$\mathbf{\Sigma}_\delta : (\mathbf{E}_\delta, \mathbf{A}_\delta, \mathbf{B}_\delta, \mathbf{C}_\delta, \mathbf{0})$$

has the same transfer function as the initial system Σ.

(c) Download the file `bandstop.mat` and use the `load` command to load its contents into MATLAB.

- (i) Take only the first input with the second output (SISO) with D=0 and sample the transfer function in $N = 100$ points inside the frequency interval $[0.1, 10]Hz$.
- (ii) Separate your measurements in *left interpolation points:* $\{\mu_i\}_{i=1}^n \subset \mathbb{C}$ with the corresponding left responses $\{\nu_i\}_{i=1}^n \in \mathbb{C}$ and in right interpolation points: $\{\lambda_\mathbf{i}\}_{\mathbf{i=1}}^{\mathbf{n}} \subset \mathbb{C}$ with responses $\{w_i\}_{i=1}^n \in \mathbb{C}$. (In the case of SISO the direction matrices are considered as $\mathbf{1}$).
- (iii) Form the Loewner matrices as:

$$\mathbb{L}_{(ij)} = \frac{\nu_i - w_j}{\mu_i - \lambda_j}, \quad \mathbb{L}_{s(ij)} = \frac{\mu_i \nu_i - w_j \lambda_j}{\mu_i - \lambda_j}, \quad i, j = 1, ..., n.$$

- (iv) Transform all the complex quantities to real under the assumption of $\bar{H}(s) = H(\bar{s})$.
- (v) Compute the singular value decomposition (SVD) of the augmented matrices $[\mathbb{L} \quad \mathbb{L}_s]$ and $[\mathbb{L}; \mathbb{L}_s]$. Decide the order of the reduced model and get the projectors $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times r}$.
- (vi) Construct the projected model as:

$$\{\hat{\mathbf{E}} = -\mathbf{Y}^*\mathbb{L}\mathbf{X}, \hat{\mathbf{A}} = -\mathbf{Y}^*\mathbb{L}_s\mathbf{X}, \hat{\mathbf{B}} = \mathbf{Y}^*\mathbf{V}, \hat{\mathbf{C}} = \mathbf{W}\mathbf{X}\}$$

- (vii) Evaluate the approximant of order r inside the interval $[0.1, 10]Hz$ with 1000 points. Superimpose the two transfer functions.

(g) What is the order r which we are able to recover the initial system?

- (i) With $D = 0$
- (ii) With $D \neq 0$

(h) Find the controllability and observability Gramian of the initial system and hence show that the system is balanced. What are the Hankel singular values σ_k, $k = 1, ..., 10$?

(i) Show that the 2-port system is all-pass if the feedthrough matrix is chosen as: $\hat{\mathbf{D}} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$.

(j) Hence conclude that this system cannot be approximated using balanced truncation.

[1] A band-stop filter is a dynamical system that blocks signals with frequency in a given interval while it lets through, almost unchanged, signals with frequency outside the given interval.

• **Closing remarks**. System matrices of a discretized beam can be downloaded the `beam.mat` file from

`http://www.slicot.org/shared/bench-data/beam.zip` ,

which contains system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ for a single-input, single-output (SISO) system with $n = 348$ states describing a `clamped beam`. Notice that $\mathbf{A}$ is stored in sparse format; however, because it has modest dimensions, we can convert it to dense format in MATLAB using the command `A = full(A)`. Alternatively, you can use another system from your own research, or one of the benchmarks available at `http://www.icm.tu-bs.de/NICONET/benchmodred.html` . • **Reference**. For more details on model

order reduction, we direct the reader to
A.C. Antoulas, *Approximation of large-scale dynamical systems*, SIAM, Philadelphia, 2005.