Dr. Maximilian Merkert
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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Compact Course Polynomial Optimization – Series 1

https://www.mathcore.ovgu.de/TEACHING/COMPACTCOURSES/2020opt.php

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Exercise 1.1

Consider the polynomial optimization problem

$$\inf\{f(x) \mid x \in \mathbb{R}^n\}$$

with $f \in \mathbb{R}[X]$.
Show that the following holds:

a) If $n = 1$ and the infimum is finite, it will be attained at some $x \in \mathbb{R}^n$.

b) For every $n \geq 2$, there exists a polynomial f such that the infimum is finite but not attained.

Exercise 1.2

Show that if a matrix $A \in \mathcal{S}^k$ is psd, then it can be written as

$$A = u_1 u_1^\top + \ldots + u_r u_r^\top$$

for finitely many vectors $u_1, \ldots, u_r \in \mathbb{R}^k$.
Can the choice of r be bounded in terms of k?

Exercise 1.3

Consider the polynomial

$$f = 2 + X_1^2 + X_1^2 X_2^4 - 4X_1 X_2.$$

a) Show that f is SOS.

b) Determine a vector $m(X)$ of monomials and a PSD matrix Z with $f = m(X)^\top Z m(X)$.

c) Describe, for your choice of $m(X)$, all PSD matrices Z satisfying $f = m(X)^\top Z m(X)$.

Exercise 1.4

Consider the homogenization

$$h(X_1, X_2, X_3) := X_3^6 - 3X_1^2 X_2^2 X_3^2 + X_1^2 X_2^4 + X_1^4 X_2^2$$

of the Motzkin polynomial (called the *Motzkin form*). Show that

a) $f = h(X_1, 1, X_3)$ is non-negative.

b) f is not SOS.

c) $h(X_1, 1, X_3) + c$ is SOS for some $c \in \mathbb{R}$.