Polynomial Optimization: Moment Relaxations.

$\underline{g} = (g_1, \dots, g_m)$, $g_i \in \mathbb{R}[X]$.

Today: $S = S(\underline{g}) = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0 \dots g_m(x) \geq 0\}$

is <u>always</u> assumed to be compact.

Final goal: understand the <u>dual</u> of the SDP program $f^*_{SOS,r} = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \in M(\underline{g})_{2r}\}$

<u>Dual SDP</u>: denote $vec(f)$ the vector of coeffs of f in the monomial basis $\left(vec(f) = \begin{pmatrix} \vdots \\ f_\alpha \\ \vdots \end{pmatrix} \text{ if } f = \sum f_\alpha x^\alpha\right)$

then the dual can be written as

$$f^*_{mom,r} := \inf \left\{ (\dots y_\alpha \dots) \begin{pmatrix} \vdots \\ f_\alpha \\ \vdots \end{pmatrix} \in \mathbb{R} \;\middle|\; \begin{pmatrix} \vdots \\ y_\alpha \\ \vdots \end{pmatrix} \in \mathbb{R}^N,\ \begin{array}{l} M_r(\underline{y}) \succeq 0 \\ M_{r_i}(g_i \cdot \underline{y}) \succeq 0 \ \forall i \\ y_0 = 1. \end{array} \right\}$$

<u>where</u>:

- $N := \dim \mathbb{R}[x]_{\leq r}$, $r_i := r - \left\lfloor \frac{\deg g_i}{2} \right\rfloor$,
- $M_r(\underline{y}) := (y_{\alpha+\beta})_{|\alpha|,|\beta| \leq r} \in \mathbb{R}^N$
- $M_{r_i}(g_i \cdot \underline{y}) := \left(\sum_\gamma g_{i\gamma} y_{\alpha+\beta+\gamma}\right)_{|\alpha|,|\beta| \leq r_i}$ $\left(g_i = \sum_\gamma g_{i\gamma} x^\gamma\right)$

$\longrightarrow$ Make the example $f = x$, $g = 1 - x^2$, $r = 2$.

Moreover (with many simplifications):
if $\mathrm{rk}(\mathcal{M}_{r-1}(\underline{y})) = \mathrm{rk}\, \mathcal{M}_r(\underline{y}) = \boxed{\kappa}$ for $\underline{y} = (y_\alpha)$: $(-y_\alpha-) \begin{pmatrix} 1 \\ f_\alpha \\ \vdots \end{pmatrix} = f^*_{mom,r}$ (i.e. $\underline{y}$ is the output of the interior point method) then we have $f^*_{mom,r} = f^*$ (finite convergence), and there are κ minimizers that can be extracted from $\mathcal{M}_r(\underline{y})$.

Question: what's happening? $\to$ Duality

Definition A Borel (a Radon) measure supported on S is a (finite, positive) measure on the σ-algebra generated by the compact subsets of S.
We denote the set of Borel measures by $\mathcal{M}(S)$.

Concrete examples:

- Dirac deltas δ_x (for $x \in S$). We have, for $A \subset S$:

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}, \quad \text{and}$$

$$\int f \, d\delta_x = f(x) \quad \forall f.$$

- Restriction of Lebesgue measure. $\mathbb{1}_S(x)\,dx = d\mu$

$\mu(A) = $ Lebesgue measure of $A \cap S$.

$$\int f \, d\mu = \int_S f(x)\, dx.$$

→ there are many other things inside $\mathcal{M}(S)$:
Gaussian measures restricted to S,
integration along curves $C \subset S$, ... We have:
$f^* = \inf \{ \int f \, d\mu \mid \mu \in \mathcal{M}(S), \int 1 \, d\mu = 1 \}$ (Ex.)

Basic property of integration ($\mu \in \mathcal{M}(S)$):
$\int af + bg \, d\mu = a \int f \, d\mu + b \int f \, d\mu$ ($a, b \in \mathbb{R}$)

→ Integration is linear!
$\mathbb{R}[x]$ is an $\mathbb{R}$-vector space, $\mathbb{R}[x]^\vee = \{ L : \mathbb{R}[x] \to \mathbb{R} \mid L \text{ is linear} \}$
We denote $L_\mu : \mathbb{R}[x] \to \mathbb{R} \in \mathbb{R}[x]^\vee$ "row vectors"
$f \mapsto \int f \, d\mu$

the moments of μ are the sequence of integration of monomials:
$$\{ \int x^\alpha d\mu \}_{\alpha \in \mathbb{N}^n} = \{ \int x_1^{\alpha_1} \cdots x_n^{\alpha_n} d\mu \}_{\alpha \in \mathbb{N}^n} =$$
$$= \{ L_\mu(x^\alpha) \}_{\alpha \in \mathbb{N}^n}$$

More generally, for $L \in \mathbb{R}[x]^\vee$ the sequence:
$\{ L(x^\alpha) \}_{\alpha \in \mathbb{N}^n} = \{ \underset{L(x^\alpha) \in \mathbb{R}}{\overset{\shortparallel}{y_\alpha}} \}_{\alpha \in \mathbb{N}}$ is called pseudo-moment sequence

→ Pseudo-moments determine uniquely L:
if $f = \sum_\alpha f_\alpha x^\alpha$, then:

We substitute the monomial x^α by the value y_α.

$$L(f) = L(\sum_\alpha f_\alpha x^\alpha) = \sum_\alpha f_\alpha L(x^\alpha) = \sum f_\alpha y_\alpha =$$
$$= (- \, y_\alpha \, -) \begin{pmatrix} | \\ f_\alpha \\ | \end{pmatrix}$$ ← In coordinates linear functionals are row vectors of pseudo-moments

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The Moment Problem: Given a sequence of real numbers $\{y_\alpha\}_{\alpha \in \mathbb{N}^n}$, $\exists \mu \in \mathcal{M}(S)$: $y_\alpha = \int x^\alpha d\mu = L_\mu(x^\alpha) \quad \forall \alpha \in \mathbb{N}^n$?

Equivalently: given $L \in \mathbb{R}[x]^\vee$, $\exists \mu \in \mathcal{M}(S)$ s.t. $L = L_\mu$?

Theorem (Haviland) Given $L : \mathbb{R}[x] \to \mathbb{R}$ linear, $\exists \mu \in \mathcal{M}(S)$: $L = L_\mu$ if and only if, $\forall f \in \mathbb{R}[x]$, $f \geq 0$ on S, $L(f) \geq 0$. "$\mathcal{M}(S) = \mathcal{P}(S)^*$"

"only if" is clear: if $f \geq 0$ on S, $L_\mu(f) = \int f d\mu \geq 0$ by definition of measure and integral.

Theorem (Putinar)

Given $L : \mathbb{R}[x] \to \mathbb{R}$ linear, $\exists \mu \in \mathcal{M}(S(\underline{g}))$: $L = L_\mu$ if and only if $\forall q \in M(\underline{g})$, $L(q) \geq 0$
if and only if $\forall f \in \mathbb{R}[x]$, $L(f^2) \geq 0$, $L(f^2 g_1) \geq 0, \dots,$ $L(f^2 g_n) \geq 0$.

Applying L to a polynomial means to substitute the monomials x^α by the values $y_\alpha = L(x^\alpha)$. Therefore:

$$L(f^2) = L\left((-\ f_\alpha\ -) \begin{pmatrix} | \\ x^\alpha \\ | \end{pmatrix} (-\ x^\alpha\ -) \begin{pmatrix} | \\ f_\alpha \\ | \end{pmatrix} \right) =$$

$$= L\left((-\ f_\alpha\ -) \begin{pmatrix} x^{\alpha+\beta} \end{pmatrix}_{\alpha,\beta \in \mathbb{N}^n} \begin{pmatrix} | \\ f_\alpha \\ | \end{pmatrix} \right)$$

$$= (-\ f_\alpha\ -) \begin{pmatrix} L(x^{\alpha+\beta}) \end{pmatrix}_{\alpha,\beta} \begin{pmatrix} | \\ f_\alpha \\ | \end{pmatrix} =$$

$$= (-\ f_\alpha\ -) \begin{pmatrix} y_{\alpha+\beta} \end{pmatrix}_{\alpha,\beta} \begin{pmatrix} | \\ f_\alpha \\ | \end{pmatrix}$$

therefore, if $M(\underline{y}) = (y_{\alpha+\beta})_{\alpha,\beta}$ is the moment matrix,

$L(f^2) \geq 0\ \forall f \iff M(\underline{y}) \succeq 0.$

In a similar way: $L(g_i f^2) \geq 0\ \forall f \iff$

$\iff M(g_i \cdot \underline{y}) \succeq 0.$

Problem: infinite matrices!

Solution: Restriction to $\mathbb{R}[x]_{\leq r}$, or equivalently, $|\alpha| \leq r$.

$\longrightarrow M_r(\underline{y}), \ldots$ truncated moment and localizing matrices.

Theorem (Curto-Fialkow)

If $\mathrm{rk}\, M_r(\underline{y}) = \mathrm{rk}\, M_{r+1}(\underline{y}) = k$, then

$y_\alpha = \int x^\alpha d\mu$ is a moment linear functional,

when μ is a (conic) sum of k Dirac deltas.

Moreover, we can extract the points associated with the Dirac deltas from $M_{r+1}(y)$

Example $(n=1)$. Assume $y_0 = 1, y_1 = 0, y_2 = 0$.

$$\left.\begin{array}{l} M_0(\underline{y}) = (y_0) = 1 \\ M_1(\underline{y}) = \begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right\} \text{same rank} = 1$$

- $y_i = \int x^i d\delta_0$, δ_0 is the Dirac delta at the origin.

The point correspont to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \ker M_1(\underline{y})$.

Other formulation:

$$f^*_{mom,r} = \inf \left\{ L(f) \in \mathbb{R} \;\middle|\; L \in \mathbb{R}[x]^\vee_{\leq 2r},\ L(M(g)_{2r}) \geq 0,\ L(1) = 1 \right\}$$

Properties

- $f^*_{mom,r} \leq f^*_{mom,r+1} \leq f^* \quad \forall r.$
- $f^*_{sos,r} \leq f^*_{mom,r} \quad \forall r.$