MathCoRe
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Graph parameters

Exercise 1

Let $G = (V, E)$ be a graph. The theta number of G is defined as

$$\chi(G) = \max_{X \in \mathcal{S}^n} \{\langle J, X\rangle : \mathrm{Tr}(X) = 1, X_{ij} = 0 \ \forall \{i,j\} \in E, X \succeq 0\}.$$

Formulate the dual SDP and show that $\chi(G)$ can be expressed by any of the following programs

1. $\chi(G) = \min_{t \in \mathbb{R}, A \in \mathcal{S}^n}\{t : tI + A - J \succeq 0, A_{ij} = 0 \, (i = j \text{ or } \{i,j\} \in \overline{E})\}$
2. $\chi(G) = \min_{t \in \mathbb{R}, B \in \mathcal{S}^n}\{t : tI - B \succeq 0, B_{ij} = 1 \, (i = j \text{ or } \{i,j\} \in \overline{E})\}$
3. $\chi(G) = \min_{t \in \mathbb{R}, C \in \mathcal{S}^n}\{t : C - J \succeq 0, C_{ii} = 1 \, (i \in V), C_{ij} = 0 \, (\{i,j\} \in \overline{E})\}$
4. $\chi(G) = \min_{B \in \mathcal{S}^n}\{\lambda_{\max}(B) : B_{ij} = 1 \, (i = j \text{ or } \{i,j\} \in \overline{E})\}$

Fact: Strong duality applies and the supremum of the primal program and the infimum of the dual program are attained.

Exercise 2

1. Show that the inequality $\omega(G) \leq \chi(G)$ between the clique and the chromatic number of a graph can be strict.
2. Show that all bipartite graphs are perfect.
3. Show that any graph G can be embedded in a graph $\hat{G}$ with $\omega(\hat{G}) = \chi(\hat{G})$. Is $\hat{G}$ necessarily perfect?

Exercise 3

Proof the inequality

$$\alpha(G) \leq \alpha^*(G) \leq \chi(\overline{G})$$

and find an example of a graph where all inequalities are strict.

Hint: Consider circles of length n. Is there a difference between n even or odd?

Quadratic modules and preorderings (if you want, you can skip to the next section about Polynomial Optimization)

Exercise 5

Prove that the quadratic module generated by $\mathbf{g}$, i.e. $M(\mathbf{g}) = \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_m$, is the smallest quadratic module containing $g_1, \ldots g_m$. Moreover, show that $M(\mathbf{g}) \subsetneq \mathbb{R}[\mathbf{x}]$ if and only if $-1 \notin M(\mathbf{g})$.

Exercise 6

Show that, if $f \in \mathbb{R}[\mathbf{x}]$, then $M(f, -f) = I_{\mathbb{R}}(f) + \Sigma^2$.

Exercise 7

We consider the univariate case.

1. Show that $M(1-x, 1+x) = T(1-x^2)$. *There is a deep reason for this equality: if $S(\mathbf{g}) \subset \mathbb{R}$ is compact, then $M(\mathbf{g}) = T(\mathbf{g})$. This is an highly non-trivial result in full generality.*
2. Prove that $(1-x)^3 \notin M((1-x)^3, 1+x)$. *Hint: you can first show that $x \notin M(x^3)$, and then adapt the proof.*

3. Prove that $x \notin M(-x^2)$, but then show that, for all $\varepsilon > 0$, $x + \varepsilon = \sigma_{0,\varepsilon} - \sigma_{1,\varepsilon} x^2 \in M(-x^2)$ with $\deg \sigma_{0,\varepsilon} = 2, \deg \sigma_{1,\varepsilon} = 1$. *Hint: use some of the previously derived equations, and then substitute $x \to x/\varepsilon$. What happens to the norm of the coefficients when $\varepsilon \to 0$?*

Exercise 8

Prove that $x_1 x_2 \notin M(x_1, x_2, 1 - x_1^2 - x_2^2)$.

Exercise 8

Let $(a_1, \dots a_n) \in \mathbb{R}^n$ and consider the ideal $I = I_{\mathbb{R}}(x_1 - a_1, \dots, x_n - a_n)$. Show that $I \subset \mathbb{R}[\mathbf{x}]$ is a maximal proper ideal (i.e. show that, if $f \in \mathbb{R}[\mathbf{x}] \setminus I$, then any ideal containing I and f is equal to $\mathbb{R}[\mathbf{x}]$).

Hint: equivalently, you can prove that the quotient ring $\mathbb{R}[\mathbf{x}]/I$ is isomorphic to $\mathbb{R}$. Notice also that the statement is true if we replace $\mathbb{R}$ by an arbitrary field.

Polynomial Optimization

Exercise 9

Show that, for $D \subset \mathbb{R}^n$:

$$f^* := \inf\{\, f(\mathbf{x}) \mid \mathbf{x} \in D \,\} = \sup\{\, \lambda \in \mathbb{R} \mid f(\mathbf{x}) - \lambda \geq 0 \ \forall \mathbf{x} \in D \,\}$$

(we are not assuming compactness of D)

Exercise 10

Find an example of quadratic module where $M(\mathbf{g})_r \subsetneq M(\mathbf{g}) \cap \mathbb{R}[\mathbf{x}]_{\leq r}$ for some r.

Hint: consider the univariate polynomial $1 - x^2$. *Show that*

$$1 - x^2 \in (M(1-x, 1+x) \cap \mathbb{R}[\mathbf{x}]_{\leq 2}) \setminus M(1-x, 1+x)_2$$

Exercise 11

Show the following basic properties of the SoS hierarchy $\{ f^*_{\text{SoS},r} \}_{r\in\mathbb{N}}$:

- $\forall r \in \mathbb{N}$, $f^*_{\text{SoS},r} \leq f^*$
- $\forall r \in \mathbb{N}$, $f^*_{\text{SoS},r} \leq f^*_{\text{SoS},r+1}$